Integrable Operators and Canonical Differential Systems

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Abstract

In this article we consider a class of integrable operators and investigate its connections with the following theories:the spectral theory of non-self-adjoint operators, the Riemann-Hilbert problem, the canonical differential systems and the random matrices theory.

Introduction

In the article [12] we considered the operators of the type

$$Sf = L(x)f(x) + P.V.\int_a^b \frac{D(x,t)}{x-t} f(t)dt, \tag{1}$$

where $f(x) \in L_k^2(a,b)$ and $k \times k$ matrix functions $L(x)$ and $D(x,t)$ are such that

$$L(x) = L^\star(x), \quad D(x,t) = -D^\star(t,x). \tag{2}$$

(The symbol P.V. indicates that the corresponding integral is understood as the principal value.)
Later in the work [8] the important class of the operators S ,when

$$k = 1, \quad L(x) = 1, \quad D(x,x) = 0, \tag{3}$$

was studied in details. These results had a number of interesting applications [5],[8].
In our works [12],[13] the connection of the operators S with the spectral theory of non-selfadjoint operators was shown.The operator identity

$$(QS - SQ)f = \int_a^b D(x,t)f(t)dt, \quad Qf = xf(x), \tag{4}$$

plays an essential role in these articles. From the identity (4) follows the statement.

Proposition 1. *Let the kernel* $D(x,t)$*be degenerate , i.e.* $D(x,t)=iA(x)A^{\star}(t)$*, where* $A(x)$ *is a* $k{\times}m$ *matrix function* ($k{\leqslant}m$)*. If the operator* S *is invertible , then the operator* $T=S^{-1}$ *has the form*

$$Tf=M(x)f(x)+P.V.\int_a^b \frac{E(x,t)}{x-t}f(t)dt, \tag{5}$$

where $M(x)=M^{\star}(x)$ *and the kernel* $E(x,t)$ *is also degenerate and has the form*

$$E(x,t)=iB(x)B^{\star}(t), \tag{6}$$

$B(x)$ *is a* $k{\times}m$ *matrix function.*

The operators S and T lead to the Riemann-Hilbert matrix problem

$$W_+(\sigma)=W_-(\sigma)R^2(\sigma), \quad a{\leqslant}\sigma{\leqslant}b, \tag{7}$$

where $m{\times}m$ matrix function $W(z)$ is analytic , when $z{\notin}[a,b]$. Here matrix function $R^2(\sigma)$ can be constructed with the help of the operators S and T, $\quad W_{\pm}(\sigma)$ is defined by the relation

$$W_{\pm}(\sigma)=\lim W(z), \quad y{\longrightarrow}0, \quad z=\sigma+iy. \tag{8}$$

In the present article an essential role is played by the canonical differential system

$$\frac{d}{dx}W(x,z)=i\frac{JH(x)}{z-x}W(x,z), \quad W(0,z)=I_m, \tag{9}$$

where $m{\times}m$ matrix J is such that

$J=J^{\star}, \quad J^2=I_m$ and $H(x){\geqslant}0$.

The monodromy matrix of system (9) coincides with the solution of the Riemann-Hilbert problem (7), i.e.

$$W(z)=W(b,z). \tag{10}$$

Let us note that $W(z)$ is a characteristic matrix function of the operator (see [2],[10])

$$Af = xf + i\int_a^x \beta(x)J\beta^{\star}(t)f(t)dt, \quad f(x){\in}L_k^2(a,b), \tag{11}$$

where $\beta(x)$ is a $k{\times}m$ matrix function such, that

$$\beta^{\star}(x)\beta(x) = H(x). \tag{12}$$

We deduce in this article a new sufficient condition of the linear similarity of the operator A to the operator $Qf = xf$. It easily follows from (9) that $W(x,z)$ in the neighborhood of $z = \infty$ admits the representation

$$W(x,z) = I_m + \frac{M_1(x)}{z} + \frac{M_2(x)}{z^2} + ..., \tag{13}$$

where

$$M_1(x) = i\int_a^x JH(t)dt. \tag{14}$$

In view of (9) and (14) all the coefficients $M_k(x)$ are defined if the coefficient $M_1(x)$ is known. This fact is of interest as the representation

$$W(b,z) = I_m + \frac{M_1(b)}{z} + \frac{M_2(b)}{z^2} + ... \tag{15}$$

is closely connected with the problems of the random matrices theory [4],[14]. From the view point of the random matrix theory it is important that in this article the procedure of constructing the matrix $M_1(x)$ is given (section 3). We pay the principal attention to the matrix version of the class (3), when

$$k{\geqslant}1, \quad L(x) = I_k, \quad D(x,x) = 0. \tag{16}$$

For this class the corresponding matrix function $R^2(x)$ from (7) has a special structure, namely

$$[R^2(x) - I_m]^2 = 0. \tag{17}$$

The corresponding matrix function $JH(x)$ is nilpotent when $m=1$, i.e.

$$[JH(x)]^2 = 0. \tag{18}$$

In the last part of the paper we consider a number of examples.

1 Integrable operators and Riemann-Hilbert problem

In this section we remind of a number of facts contained in the paper [12]. We use these facts in the next sections. Let $W(z)$ be $m{\times}m$ matrix function.
We suppose that the following conditions are fulfilled.
1). Matrix function $W(z)$ is analytic in the domain $z{\notin}[a,b], (-\infty < a < b < \infty)$ and satisfies the equality

$$W(z) = I_m + \frac{1}{2\pi i}\int_a^b \frac{F(x)}{x-z}dx, \tag{19}$$

where $F(x)$ is bounded $m{\times}m$ matrix function on the segment [a,b].
2). The relations

$$W^{\star}(z)JW(\bar{z}) = J, \tag{20}$$

$$i\frac{W^{\star}(z)JW(z)-J}{z-\bar{z}}{\geqslant}0, \quad z{\neq}\bar{z} \tag{21}$$

are true. (Here $m{\times}m$ matrix J satisfies the equalities $J = J^{\star}, \quad J^2 = I$).
The equality (1) guarantees the almost everywhere existence of the limits

$$W_{\pm}(x) = lim W(z) \quad as \quad y{\rightarrow}{\pm}0, \quad z = x+iy. \tag{22}$$

Now we use the polar decomposition (see [11])

$$W_{+}(x) = U(x)R(x), \quad W_{-}(x) = U(x)R^{-1}(x), \tag{23}$$

where $m\times m$ matrix functions $U(x)$ and $R(x)$ are such that

$$U^{\star}(x)JU(x)=J, \quad JR(x)=R^{\star}(x)J \tag{24}$$

and in addition the spectrum of $R(x)$ is positive.
Matrix function $R(x)$ is called *J-module* of matrix function $W_{+}(x)$. By relations (23) and (24) we have

$$R^2(x)=JW_{+}^{\star}(x)JW_{+}(x). \tag{25}$$

According to the theory of J-module [11] the relations

$$D(x)=J[R(x)-R^{-1}(x)]\geqslant 0, \quad x\in[a,b], \tag{26}$$

$$D(x)=0, \quad x\notin[a,b] \tag{27}$$

are true. Now we introduce the matrix functions $F_1(x), F_2(x)$ with the help of the relations

$$F_1^{\star}(x)F_1(x)=D(x), \quad F_2(x)=F_1(x)JU^{\star}(x). \tag{28}$$

Remark 1. Matrix functions $F_1(x)$ and $F_2(x)$ are $k\times m$ matrices, where $k=\sup[\ rank D(x)], a\leqslant x\leqslant b$. Hence $k\leqslant m$.
Using relations (23),(26) and (28) we can write

$$W_{+}(x)-W_{-}(x)=F_2^{\star}(x)F_1(x)=F(x). \tag{29}$$

In addition to conditions 1) and 2) we suppose:
3). The matrix functions $F_1(x)$ and $F_2(x)$are bounded on segment [a,b].
Let us define the operators Π and Γ by formulas $\Pi g=\frac{1}{\sqrt{2\pi}}F_1(x)g$,
$\Gamma g=-\frac{i}{\sqrt{2\pi}}F_2(x)g$, where g are $m\times 1$ vectors, Πg and Γg belong to $L_k^2(a,b)$. Then we have

$$\Pi^{\star}f(x)=\frac{1}{\sqrt{2\pi}}\int_a^b F_1^{\star}(x)f(x)dx, \tag{30}$$

$$\Gamma^{\star} f(x) = \frac{i}{\sqrt{2\pi}} \int_a^b F_2^{\star}(x) f(x) dx, \tag{31}$$

where $f(x) \in L_k^2(a,b)$. The next assertion follows from formulas (19) ,(30) and (31).

Proposition 2. *The matrix function W(z) admits the realization*

$$W(z) = I_m - \Gamma^{\star}(Q - zI)^{-1}\Pi, \tag{32}$$

where the operator Q is defined by the relation

$$Qf = xf, \quad f(x) \in L_k^2(a,b). \tag{33}$$

Next we introduce the $k \times k$ matrix

$$L(x) = [I_k + \frac{1}{4}(F_1(x) J F_1^{\star}(x))^2]^{1/2} \tag{34}$$

and consider the operators

$$Sf = L(x) f(x) + \frac{i}{2\pi} P.V. \int_a^b \frac{F_1(x) J F_1^{\star}(t)}{x - t} f(t) dt, \tag{35}$$

$$Tf = L(x) f(x) - \frac{i}{2\pi} P.V. \int_a^b \frac{F_2(x) J F_2^{\star}(t)}{x - t} f(t) dt. \tag{36}$$

The introduced operators S and T are acting in the space $L_k^2(a,b)$ and $f(x)$ is a $k \times 1$ vector function.

Theorem 1.(see [13], p.45-46) *The operators S and T are positive , bounded and*

$$T = S^{-1}, \quad SF_2(x) = F_1(x) J. \tag{37}$$

From relation (23) we deduce that

$$W_+(x) = W_-(x) R^2(x), \quad x \in [a,b] \tag{38}$$

$$W_+(x) = W_-(x), \quad x \notin [a, b] \tag{39}$$

Formulas (38) and (39) lead to the Riemann-Hilbert Problem.

Problem 1. *To recover the matrix function $W(z)$ by the given J-module $R(x)$.*

In the case $J = I$ Problem 1 plays an essential role in the prediction theory of the stationary processes [15]. The case when $J \neq I$ is important for the theory of random matrices [5], [8],[14].

We solve Problem 1 in the following way.

1.By the given matrix $R^2(x)$ we construct the matrix $D(x)$ (see (26)).

2.Using the first of equalities (28) we find $F_1(x)$.

3.With the help of formula (1) the operator S is constructed.

4.Due to the second equality of (37) we have $F_2(x) = S^{-1}F_1(x)J$.

5.Now it is easy to see that formulas (19) and (29) give the solution of the Riemann-Hilbert problem (7) with the normalizing condition

$$W(z) \to I \quad as \quad z \to \infty. \tag{40}$$

Remark 2. The operators S and T defined by formulas (1) and (5) are called integrable [5], [8].The case when $k = 1$ and

$$F_1(x)JF_1^{\star}(x) = 0 \tag{41}$$

has important applications in the theory of the random matrices(see [4], [7], [8], [14]). The general case was used in the spectral theory of the non-selfadjoint operators [12],[13].

2 Spectral theory

We introduce some important notions .

Let the linear bounded operator have the form

$$A = A_R + iA_I, \tag{42}$$

where A_R and A_I are self-adjoint operators acting in Hilbert space H.There is a bounded linear operator K which maps a Hilbert space G in H so that

$$A_I = KJK^\star, \tag{43}$$

where J acts in G and $J = J^\star, \quad J^2 = I$.

Definition 2 (see [2], [10]).*The operator function*

$$W(\lambda) = I - 2iK^\star(A - \lambda I)^{-1}KJ \tag{44}$$

is called the characteristic operator function of A.

We recall that the simple part of A means the operator which is induced by A on the subspace $H_1 = \overline{\sum_{k=0}^{\infty} A^k D_A}$, where $D_A = \overline{(A - A^\star)H}$. In paper [12] we deduced Theorem 1 for the case $m\leqslant\infty$. From this fact we obtain the following assertion [12],[13].

Theorem 2. *If the characteristic operator function $W(z)$ of the operator A satisfies the condition*

$$||W(z)||\leqslant c, \quad z\neq\bar{z} \tag{45}$$

for some c , then the simple part of A is linearly similar to a self-adjoint operator with a absolutely continuous spectrum

It follows from relation (45) that $W(z)$ satisfies the conditions 1)-3).The converse is not true. Using this fact we receive a new version of Theorem 2.

Theorem 3. *If the characteristic operator function $W(z)$ of the operator A satisfies the conditions 1)-3) , then the statement of Theorem 2 is true.*

Example. We consider the case when

$$F_1(x) = [x+i, x-i], \quad 0\leqslant x\leqslant 1, \quad j = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{46}$$

. The corresponding operator S has the form

$$Sf = f(x) - \frac{1}{\pi}\int_0^1 f(t)dt. \tag{47}$$

Due to relations (46) and (47) we have

$$F_2(x) = [-q(x), \overline{q(x)}], \tag{48}$$

where

$$q(x) = x + \frac{1}{2(\pi-1)} + i\frac{\pi}{\pi-1}. \tag{49}$$

Using the property of the Cauchy integral (see[6]) we deduce from relation (19) that

$$W(z) = -\frac{1}{2\pi i}F(0)logz + 0(1), \quad z\neq\bar{z}, \quad |z| < \frac{1}{2}, \tag{50}$$

$$W(z) = -\frac{1}{2\pi i}F(1)log(z-1) + 0(1), \quad z\neq\bar{z}, \quad |z-1| < \frac{1}{2}. \tag{51}$$

It follows from formulas (46) and (48),(49) that $F(0)\neq 0, \quad F(1)\neq 0$. Hence the constructed $W(z)$ satisfies the conditions of Theorem 3 but does not satisfy the condition (45) of Theorem 2.

3 Canonical differential systems

It follows from Theorem 3 that the following operator

$$S_\xi f = L(x)f(x) + \frac{i}{2\pi}P.V.\int_a^\xi \frac{F_1(x)JF_1^*(t)}{x-t}f(t)dt \tag{52}$$

is positive , bounded and invertible.

We set

$$\Phi(\xi,x)=S_{\xi}^{-1}F_1(x), \tag{53}$$

$$B(\xi)=\frac{1}{2\pi}\int_a^{\xi}\Phi^{*}(\xi,x)F_1(x)dx. \tag{54}$$

Lemma 1. *The matrix function $B(\xi)$ is absolutely continuous and monotonically increasing .*

Proof. As it is known [3],[9] the operator S^{-1} can be represented in the form

$$S^{-1}=U^{\star}U, \tag{55}$$

where the linear bounded operator U acts in the space $L_k^2(a,b)$ and satisfies the condition

$$U^{\star}P_{\xi}=P_{\xi}U^{\star}P_{\xi}, \quad a\leqslant\xi\leqslant b, \tag{56}$$

where $P_{\xi}f(x)=f(x), a\leqslant x\leqslant\xi$ and $P_{\xi}f(x)=0, \quad \xi\leqslant x$. From relations (54) and (55) we deduce the equality

$$\frac{d}{dx}B(x)=H(x)=\frac{1}{2\pi}h^{\star}(x)h(x), \tag{57}$$

where

$$h(x)=UF_1(x). \tag{58}$$

The lemma is proved.

Let us consider the system of equations

$$W(x,z)=I+iJ\int_a^x\frac{dB(\xi)}{z-\xi}W(\xi,z). \tag{59}$$

Theorem 4. (see[13], Ch.3) *The following equality*

$$W(b,z)=W(z) \tag{60}$$

holds.

Corollary 1. *The integral system (59) is equivalent to the differential system*

$$\frac{dW(x,z)}{dx} = \frac{iJH(x)}{z-x} W(x,z), \quad H(x) \geqslant 0 \tag{61}$$

with the boundary condition $W(a,z) = I_m$. *Here the matrix function H(x) is defined by relation (57) .*

Corollary 2. *The matrix function* $W(z)$ *is the monodromy matrix of system (61), i.e.* $W(z) = W(b,z)$.

Due to (61) in the neighborhood of $z = \infty$ the following relation

$$W(x,z) = I + M_1(x)/z + M_2(x)/z^2 + ... \tag{62}$$

is fulfilled. It follows from (59) and (61) that

$$M_1(x) = iJB(x). \tag{63}$$

Formulas (53) (54) and (63) give the solution of the following inverse problem.

Problem 2. *To recover the matrix function* $H(x)$ *and* $M_1(x)$ *by the given J-module* $R(x)$. Theorem 3 and relation (54) imply the following assertion.

Proposition 3. *If equality*

$$F_1(x) = 0, \quad \alpha \leqslant x \leqslant \beta, \quad \alpha \neq \beta \tag{64}$$

is true then

$$F_2(x) = 0, \quad W_+(x) = W_-(x), \quad R(x) = I, \quad \alpha \leqslant x \leqslant \beta. \tag{65}$$

Corollary 3. *If condition (64) is fulfilled then*

$$B'(x) = H(x) = 0, \quad \alpha \leqslant x \leqslant \beta. \tag{66}$$

4 Examples

Example 1. Let us consider the case when

$$J = j = \begin{bmatrix} -I_m & 0 \\ 0 & I_m \end{bmatrix} \tag{67}$$

and

$$R^2(x) = \begin{bmatrix} 0 & \phi(x) \\ -\phi^\star(x) & 2I_m \end{bmatrix}, \quad 0 \leqslant x \leqslant r, \tag{68}$$

where $\phi(x)\phi^\star(x) = I_m$. From (68) we deduce that

$$R(x) = 1/2 \begin{bmatrix} I_m & \phi(x) \\ -\phi^\star(x) & 3I_m \end{bmatrix} \tag{69}$$

The matrix $R(x)$ satisfies the following conditions.

1. *The spectrum of $R(x)$ is positive.*

Indeed, we obtain by direct calculation that $[R(x)-I]^2 = 0$. Hence the spectrum of the matrix $R(x)$ is concentrated at the point $\lambda = 1$.

2. *The relation*

$$jR(x) = R^\star(x)j \tag{70}$$

is true.

It means that $R(x)$ is the j-module of the matrix $W(z)$ which satisfies relation (7). From (68) we deduce that

$$R(x) - R^{-1}(x) = \begin{bmatrix} -I_m & \phi(x) \\ -\phi^\star(x) & I_m \end{bmatrix}. \tag{71}$$

According to (71) we have

$$D(x) = j[R(x) - R^{-1}(x)] = \begin{bmatrix} I_m & -\phi(x) \\ -\phi^\star(x) & I_m \end{bmatrix}. \tag{72}$$

Hence the equality

$$F_1(x) = [I_m, -\phi(x)] \tag{73}$$

holds. Using (73) we obtain the relations

$$F_1(x)jF_1^{\star}(x) = 0, \tag{74}$$

$$F_1(x)jF_1^{\star}(t) = \phi(x)\phi^{\star}(t) - I_m \tag{75}$$

Thus in case (69) we deduce from (52) and (74) ,(75) that operator the S_ξ has the form

$$S_\xi f = f(x) + \frac{i}{2\pi} P.V. \int_0^\xi \frac{\phi(x)\phi^{\star}(t) - I_m}{x-t} f(t)dt. \tag{76}$$

The fact that the operator V defined as

$$Vf = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt, \quad f \in L^2(-\infty, \infty) \tag{77}$$

is unitary implies that

$$S_\xi \geqslant 0. \tag{78}$$

Further we suppose that the operator S_r is invertible.

Hence the operators $S_\xi, \quad \xi \leqslant r$ are invertible as well.

Remark 3. If $\phi(x)$ satisfies Hölder condition then there exists such $r > 0$ that S_r is invertible.

Using relation (53) we have

$$\Phi(x,\xi) + \frac{i}{2\pi} P.V. \int_0^\xi \frac{\phi(x)\phi^{\star}(t) - I_m}{x-t} \Phi(t,\xi)dt = F_1(x). \tag{79}$$

where

$$\Phi(x,\xi) = [\Phi_1(x,\xi), \Phi_2(x,\xi)]. \tag{80}$$

Here $\Phi_k(x,\xi)$ are $m\times m$ matrix functions $(k = 1,2)$. It follows directly from (73) and (79) that

$$\Phi_1(x,\xi) + \frac{i}{2\pi} P.V. \int_0^{\xi} \frac{\phi(x)\phi^{\star}(t) - I_m}{x-t} \Phi_1(t,\xi)dt = I_m, \tag{81}$$

$$\Phi_2(x,\xi) + \frac{i}{2\pi} P.V. \int_0^{\xi} \frac{\phi(x)\phi^{\star}(t) - I_m}{x-t} \Phi_2(t,\xi)dt = -\phi(x), \tag{82}$$

and

$$\Phi_1(x,\xi)\Phi_1^{\star}(x,\xi) = \Phi_2(x,\xi)\Phi_2^{\star}(x,\xi). \tag{83}$$

Due to (37) and (54) the formulas

$$F_2(x) = [-\Phi_1(x,1), \Phi_2(x,1)], \tag{84}$$

$$B(\xi) = \frac{1}{2\pi} \int_0^{\xi} \begin{bmatrix} \Phi_1(x,\xi) & \Phi_2(x,\xi) \\ -\phi^{\star}(x)\Phi_1(x,\xi) & -\phi^{\star}(x)\Phi_2(x,\xi) \end{bmatrix} dx. \tag{85}$$

are true.

Example 2. We separately consider the partial case of Example 1 , when $m = 1$.

It follows from (72) and (82) that

$$\Phi_2(x,\xi) = -\phi(x)\overline{\Phi_1(x,\xi)}. \tag{86}$$

Hence formula (85) takes the form:

$$B(\xi) = \frac{1}{2\pi} \int_0^{\xi} \begin{bmatrix} \Phi_1(x,\xi) & -\overline{\Phi_1(x,\xi)}\phi(x) \\ -\overline{\phi(x)}\Phi_1(x,\xi) & \overline{\Phi_1(x,\xi)} \end{bmatrix} dx. \tag{87}$$

Comparing formulas (57) and (87) we deduce the representation

$$H(x) = B'(x) = a(x) \begin{bmatrix} 1 & e^{i\alpha(x)} \\ e^{-i\alpha(x)} & 1 \end{bmatrix}, \tag{88}$$

where $a(x)\geqslant 0, \quad \alpha(x) = \overline{\alpha(x)}$. Due to (88) the matrix $jH(x)$ is nilpotent, i.e.

$$[jH(x)]^2 = 0. \tag{89}$$

Example 3. Let us consider the partial case of Example 1, when

$$m = 1, \quad \phi(x) = e^{2iux}, \quad u = \bar{u}. \tag{90}$$

Example 3 plays an important role in the theory of the random matrices [4],[7],[14]. Now the operator S_ξ takes the form

$$S_\xi f = f(x) - \frac{1}{\pi}\int_0^\xi e^{iu(x-t)}\frac{sinu(x-t)}{x-t}f(t)dt. \tag{91}$$

The operator S_ξ defined by formula (52) is invertible for all $0 < \xi < \infty$ (see [4], p.167).

We denote by $\Psi(x,\xi,u)$ the solution of the equation

$$\Psi(x,\xi,u) - \frac{1}{\pi}\int_0^\xi \frac{sinu(x-t)}{x-t}\Psi(t,\xi,u)dt = e^{-iux}. \tag{92}$$

Then according to relations (81) and (82) we have

$$\Phi_1(x,\xi,u) = e^{iux}\Psi(x,\xi,u), \quad \Phi_2(x,\xi,u) = -e^{-iux}\overline{\Psi(x,\xi,u)}. \tag{93}$$

It follows from (87) and (93), that

$$B(\xi,u) = \frac{1}{2\pi}\int_0^\xi \begin{bmatrix} e^{iux}\Psi(x,\xi,u) & -e^{iux}\overline{\Psi(x,\xi,u)} \\ -e^{-iux}\Psi(x,\xi,u) & e^{-iux}\overline{\Psi(x,\xi,u)} \end{bmatrix} dx. \tag{94}$$

Example 4. Let us consider the case when $m = 1$ and

$$J = j = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{95}$$

and

$$R(x) = \frac{1}{2}\begin{bmatrix} 2-|\psi(x)|^2 & -\overline{\psi(x)}^2 \\ \psi(x)^2 & 2+|\psi(x)|^2 \end{bmatrix}, \quad 0\leqslant x\leqslant r. \tag{96}$$

The matrix $R(x)$ satisfies the following conditions.

1. *The spectrum of $R(x)$ is positive.*

Indeed, we obtain by direct calculation that $[R(x)-I]^2 = 0$. Hence the spectrum of the matrix $R(x)$ is concentrated at point $\lambda = 1$.

2. *The relation*

$$jR(x) = R^{\star}(x)j \tag{97}$$

is true.

It means that $R(x)$ is the j-module of the matrix $W(z)$ which satisfies relation (7). From (96) we deduce that

$$R(x) - R^{-1}(x) = jD(x) = jF_1^{\star}(x)F_1(x), \tag{98}$$

where

$$F_1(x) = [\psi(x), \overline{\psi(x)}]. \tag{99}$$

Using (99) we obtain the relations

$$F_1(x)jF_1^{\star}(x) = 0, \tag{100}$$

$$F_1(x)jF_1^{\star}(t) = \psi^{\star}(x)\psi(t) - \psi(x)\psi^{\star}(t) \tag{101}$$

Thus we deduce from (99) and (101) , that the operator S_ξ in case (96) has the form

$$S_\xi f = f(x) + \frac{i}{2\pi} P.V.\int_0^{\xi} \frac{\psi^{\star}(x)\psi(t) - \psi(x)\psi^{\star}(t)}{x-t} f(t)dt. \tag{102}$$

Further we suppose that the operators S_ξ are positive and invertible $(0 < \xi\leqslant r)$.

Remark 4. If $\psi(x)$ satisfies the Hölder condition, then there exists such $r > 0$

that the operators S_ξ are positive and invertible $(0 < \xi \leqslant r)$.

It follows directly from (99) and (102), that

$$\Phi_1(x,\xi) + \frac{i}{2\pi} P.V. \int_0^\xi \frac{\psi^\star(x)\psi(t) - \psi(x)\psi^\star(t)}{x-t} \Phi_1(t,\xi)dt = \psi(x), \tag{103}$$

$$\Phi_2(x,\xi) + \frac{i}{2\pi} P.V. \int_0^\xi \frac{\psi^\star(x)\psi(t) - \psi(x)\psi^\star(t)}{x-t} \Phi_2(t,\xi)dt = \overline{\psi(x)}, \tag{104}$$

where

$$\Phi_1(x,\xi)\Phi_1^\star(x,\xi) = \Phi_2(x,\xi)\Phi_2^\star(x,\xi). \tag{105}$$

Due to (103) and (104) we have

$$F_2(x) = [-\Phi_1(x,1), \Phi_2(x,1)], \quad \Phi_1(x,\xi) = \overline{\Phi_2(x,\xi)} \tag{106}$$

$$B(\xi) = \frac{1}{2\pi} \int_0^\xi \begin{bmatrix} \overline{\psi(x)}\Phi_1(x,\xi) & \overline{\psi(x)\Phi_1(x,\xi)} \\ \psi(x)\Phi_1(x,\xi) & \psi(x)\overline{\Phi_1(x,\xi)} \end{bmatrix} dx. \tag{107}$$

It follows from (107) that relation (89) is true in this case as well.

Remark 5. Comparing formulas (69) and (96) we see that Examples 1 and 3 coincide when $m = 1$ and

$$\phi(x) = -\overline{\psi(x)}^2, \quad |\phi(x)| = 1. \tag{108}$$

Remark 6. If

$$\psi(x) = i\sqrt{\gamma}e^{-iux}, \quad 0 < \gamma \leqslant 1, \tag{109}$$

due to (96) we have

$$R^2(x) = \begin{bmatrix} 1-\gamma & \gamma e^{2iux} \\ -\gamma e^{-2iux} & 2+\gamma \end{bmatrix}. \tag{110}$$

The corresponding Riemann-Hilbert problem was considered in [4].

Let us represent $\psi(x)$ in the form

$$\psi(x) = A(x) + iB(x), \tag{111}$$

where

$$A(x) = \overline{A(x)}, \quad B(x) = \overline{B(x)}. \tag{112}$$

Then the operator S_ξ takes the form

$$S_\xi f = f(x) - \frac{1}{\pi} P.V. \int_0^\xi \frac{A(x)B(t) - B(x)A(t)}{x - t} f(t) dt. \tag{113}$$

The following partial cases of $\psi(x)$ play an essential role in a number of applications [7]:

$$\psi_1(x) = \sqrt{\pi}[Ai(x) + iAi\prime(x)], \tag{114}$$

where $Ai(x)$ is the Airy function, and

$$\psi_2(x) = \sqrt{\frac{\pi}{2}}[J_\alpha(\sqrt{x}) + i\sqrt{x}J'_\alpha(\sqrt{x})], \tag{115}$$

where $J_\alpha(z)$ is the Bessel function.

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